J Glob Optim (2007) 37:63–73
DOI 10.1007/s10898-006-9036-x

ORIGINAL PAPER

Nonempty intersection theorems and generalized multi-objective games in product *FC*-Spaces

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Received: 10 December 2005 / Accepted: 21 April 2006 /
Published online: 6 July 2006

Abstract A new class of generalized multi-objective games is introduced and studied in *FC*-spaces where the number of players may be finite or infinite, and all payoff are all set-valued mappings and get their values in a topological space. By using an existence theorems of maximal elements for a family of set-valued mappings in product *FC*-spaces due to author, some new nonempty intersection theorems for a family of set-valued mappings are first proved in *FC*-spaces. As applications, some existence theorems of weak Pareto equilibria for the generalized multi-objective games are established in noncompact *FC*-spaces. These theorems improve, unify and generalize the corresponding results in recent literatures.

Keywords Maximal element theorem · Nonempty intersection theorem · Generalized multi-objective game · Weak Pareto equilibria · *FC*-space

1 Introduction

In recent years, much attention has been focused on the game problems with vector payoffs in game theory. One of the reasons is that multi-criteria models can be better applied to real-world situations. The motivation for the study of multi-criteria models can be found in Szidarovszky et al. [1], Zeleny [2], Bergstresser and Yu [3], and Borm et al. [4]. The existence of Pareto equilibria is one of fundamental problems. Hence, in order to guarantee the existence of (weak) Pareto equilibria of the multi-objective games, some sufficient conditions have been given by several authors (e.g., see Aubin [5], Wang [6, 7], Ding [8], Tan et al. [9], Yuan and Tarafdar [10], and Yu and Yuan [11]).

We note that, in the above models of multi-objective games, all payoffs are single-valued and take their values in finite-dimensional spaces, and most of all restrict the number of players to a finite number. Recently Guillerme [12] and Luo [13] studied

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multi-objective games with multi-valued payoffs in compact and convex setting of Hausdorff topological vector spaces. But in some case, the number of players may be infinite and for each player, the strategy set may not be convex and the payoff function may be one collection of things. Hence the study of a vector equilibrium problem for a multi-objective game with infinite players and multi-valued payoff functions in general topological spaces without any convexity structure has important applications in the real world.

Motivated and inspired by the above research works, we shall introduce and study a class of generalized multi-objective games with infinite players and with multi-valued payoffs in general topological spaces without any convexity structure.

Let I be any (finite or infinite) index set and for each $i \in I$, X_i be a topological space. We use the the following notations

$$X = \prod_{i \in I} X_i \qquad \text{and} \quad X^i = \prod_{j \in I, j \neq i} X_j.$$

For each $x \in X$, x_i and x^i denote the projection of x on X_i and X^i, respectively. Write $x = (x^i, x_i)$.

Let I be the set of players. Each player $i \in I$ has a strategy set X_i, a set-valued payoff $F_i\colon X^i \times X_i \to 2^{Z_i}$, where Z_i is a topological space and the rule of optimizing judgement $D_i\colon X \to 2^{Z_i}$ is also a set-valued mapping. A generalized multi-objective game (in short, GMOG) $\Gamma = (X_i, F_i)_{i \in I}$ is a family of ordered pair (X_i, F_i). A point $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ is said to be a weak Pareto equilibrium point of Γ if for each $i \in I$, there exists a point $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ such that

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i. \tag{1}$$

Remark 1.1 Equivalently, we can say that a point $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ is a weak Pareto equilibrium point of Γ if for each $i \in I$, there exists a point $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ such that

$$\hat{z}_i - z_i \notin D_i(\hat{x}), \quad \forall\, z_i \in F_i(\hat{x}^i, y_i) \quad \text{and} \quad y_i \in X_i. \tag{2}$$

Special cases:

(1) For each $i \in I$, if Z_i is a topological vector space and $C_i\colon X \to 2^{Z_i}$ be a set-valued mapping such that for each $x \in X$, $C_i(x)$ is a closed convex pointed cone with $\text{int}C_i(x) \neq \emptyset$, then let $D_i(x) = -\text{int}C_i(x)$ for each $i \in I$ and $x \in X$, the GMOG (1) reduces to the following GMOG $\Gamma = (X_i, F_i)_{i \in I}$: a point $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ is said to be a weak Pareto equilibrium point of Γ if for each $i \in I$, there exists a point $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ such that

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap (-\text{int}C_i(\hat{x})) = \emptyset, \quad \forall\, y_i \in X_i. \tag{3}$$

The GMOG (3) was introduced and studied by Guillerme [12] and Luo [13] in compact setting of Hausdorff topological vector spaces.

(2) For each $i \in I$, if $F_i = f_i$ is a single-valued mapping, $Y_i = \mathbf{R}^{k_i}$, where k_i is a positive integer, and $C_i(x) = \mathbf{R}^{k_i}_+ = \{u = (u_1, u_2, \ldots, u_{k_i}) \in \mathbf{R}^{k_i} : u_j \geq 0,\ \forall\, j = 1, 2, \ldots, k_i\}$, then the GMOG (3) reduce to the following multi-objective game (MOG) $\Gamma = (X_i, f_i)_{i \in I}$: a point $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ is said to be a weak Pareto equilibrium point of Γ if for each $i \in I$,

$$f_i(\hat{x}^i, \hat{x}_i) - f_i(\hat{x}^i, y_i) \notin -\text{int}C_i(\hat{x}), \quad \forall\, y_i \in X_i. \tag{4}$$

The MOG (4) includes the MOGs considered in [5–11] as special cases.

In this paper, by using an existence theorems of maximal elements for a family of set-valued mappings in product *FC*-spaces due to author, we first establish some new nonempty intersection theorems for a family of set-valued mappings. As applications, some existence theorems of weak Pareto equilibria for the GMOG (1) are established in noncompact *FC*-spaces. These theorems improve, unify and generalize the corresponding results in recent literatures.

2 Preliminaries

Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X\rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

The following notion of a *FC*-space was introduced by Ding [14, 15].

Definition 2.1 $(Y, \{\varphi_N\})$ is said to be a *FC*-space if Y is a topological space and for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N\colon \Delta_n \to Y$. A subset D of $(Y, \{\varphi_N\})$ is said to be a *FC*-subspace of Y if for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and for each $\{y_{i_0}, \ldots, y_{i_k}\} \subseteq N \bigcap D$, $\varphi_N(\Delta_k) \subseteq D$ where $\Delta_k = \mathrm{co}(\{e_{i_j} : j = 0, \ldots, k\})$.

It is easy to see that the class of *FC*-spaces includes the classes of convex sets in topological vector spaces, convex spaces [16], C-spaces (or H-spaces) [17], *G*-convex spaces [18, 19], *L*-convex spaces [20], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in *FC*-spaces. Clearly, each *FC*-subspace of a *FC*-space is also a *FC*-space.

Let X be a topological space and (Y, φ_N) be a *FC*-space. The class $\mathcal{B}(Y, X)$ of better admissible mappings was introduced by Ding [14] as follows: $F \in \mathcal{B}(Y, X) \Longleftrightarrow F\colon Y \to 2^X$ is a upper semi-continuous set-valued mapping with compact values such that for any $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and any continuous mapping $\psi\colon F(\varphi_N(\Delta_n)) \to \Delta_n$, the composition mapping $\psi \circ F|_{\varphi_N(\Delta_n)} \circ \varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point.

Clearly the our class $\mathcal{B}(Y, X)$ of better admissible mappings includes many classes of set-valued mappings as true subclasses (see [14]).

Lemma 2.1 [14] *Let I be any index set. For each $i \in I$, let $(Y_i, \{\varphi_{N_i}\})$ be a FC-space. Let $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$ where $N_i = \pi_i(N)$ is the projection of N on Y_i. Then $(Y, \{\varphi_N\})$ is also a FC-space.*

In the following, we shall assume that all topological spaces are Hausdorff. The following result is Theorem 2.8 of Ding [15].

Lemma 2.2 *Let X be a topological space, K be a nonempty compact subset of X and I be any (finite or infinite) index set. For each $i \in I$, let (Y_i, φ_{N_i}) be a FC-space and $Y = \prod_{i\in I} Y_i$ be a FC-space defined as in Lemma 2.1. Let $F \in \mathcal{B}(Y, X)$ and for each $i \in I$, $A_i\colon X \to 2^{Y_i}$ be such that for each $i \in I$,*

(1) *for each $N = \{y_0, \ldots, y_n\} \in \langle Y\rangle$ and each $M = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, $F(\varphi_N(\Delta_k)) \bigcap (\bigcap_{y\in M} A_i^{-1}(\pi_i(y))) = \emptyset$,*

(2) *for each* $y_i \in Y_i$, $A_i^{-1}(y_i)$ *is open in* X,
(3) *for each* $N_i \in \langle Y_i \rangle$, *there exists a nonempty compact FC-subspace* L_{N_i} *of* Y_i *containing* N_i *and for each* $x \in X \backslash K$, *there exists* $i \in I$ *satisfying* $L_{N_i} \bigcap A_i(x) \neq \emptyset$.

Then there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$, for each $i \in I$.

Remark 2.1 Lemma 2.2 generalizes Theorem 2.6 of Ding [21] from G-convex spaces to FC-spaces. Compare Lemma 2.2 with Theorem 2.3 of Ding [14], the condition (3) of Theorem 2.3 in [14] is removed, but the assumption that A_i^{-1} is transfer compactly open-valued is replaced by that A_i^{-1} is open-valued. Clearly, Lemma 2.2 is more convenient and useful for applications.

3 Nonempty intersection theorem

The following nonempty intersection theorem is an equivalent form of Lemma 2.2.

Theorem 3.1 *Let* X *be a topological space,* K *be a nonempty compact subset of* X *and* I *be any index set. For each* $i \in I$, *let* (Y_i, φ_{N_i}) *be a FC-space and* $Y = \prod_{i\in I} Y_i$ *be the FC-space defined as in Lemma* 2.1. *Let* $F \in \mathcal{B}(Y, X)$ *and* $G_i \colon Y_i \to 2^X$ *be such that for each* $i \in I$,

(1) *for each* $N = \{y_0, \ldots, y_n\} \in <Y>$ *and each* $M = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, $F(\varphi_N(\Delta_k)) \subseteq \bigcup_{y\in M} G_i(\pi_i(y)))$,
(2) *for each* $y_i \in Y_i$, $G_i(y_i)$ *is closed in* X,
(3) *for each* $N_i \in \langle Y_i \rangle$, *there exists a nonempty compact FC-subspace* L_{N_i} *of* Y_i *containing* N_i *and for each* $x \in X \setminus K$, *there exists* $i \in I$ *satisfying* $L_{N_i} \bigcap (Y_i \setminus G_i^{-1}(x)) \neq \emptyset$.

Then we have

$$K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i \in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Proof Lemma 2.2 $\Rightarrow$ Theorem 2.1. For each $i \in I$, define a set-valued mapping $A_i \colon X \to 2^{Y_i}$ by

$$A_i(x) = Y_i \backslash G_i^{-1}(x), \quad \forall\ x \in X.$$

Then we have that for each $y \in Y$,

$$\begin{aligned} A_i^{-1}(\pi_i(y)) &= \{x \in X : \pi_i(y) \in A_i(x)\} = \{x \in X : \pi_i(y) \in (Y_i \setminus G_i^{-1}(x))\} \\ &= \{x \in X : x \notin G_i(\pi_i(y))\} = X \backslash G_i(\pi_i(y)). \end{aligned}$$

By (1), we have that for each $N = \{y_0, \ldots, y_n\}$ and $M = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$,

$$\emptyset = F(\varphi_N(\Delta_k)) \bigcap \left(X \backslash \bigcup_{y\in M} G_i(\pi_i(y)) \right) = F(\varphi_N(\Delta_k)) \bigcap \left(\bigcap_{y\in M} (X \backslash G_i(\pi_i(y))) \right)$$

the condition (1) of Lemma 2.2 is satisfied. By (2), $A_i^{-1}(y_i) = X \backslash G_i(y_i)$ is open in X and hence the condition (2) of Lemma 2.2 is satisfied. Clearly the conditions (3) implies the condition (3) of Lemma 2.2 holds. By Lemma 2.2, there exists $\hat{x} \in K$ such that $A_i(\hat{x}) = \emptyset$ for all $i \in I$. Hence we have $A_i(\hat{x}) = Y_i \setminus G_i^{-1}(\hat{x}) = \emptyset$ and $Y_i = G_i^{-1}(\hat{x})$ for all $i \in I$. It follow that $\hat{x} \in G_i(y_i)$ for all $y_i \in Y_i$ and $i \in I$ and $\hat{x} \in \bigcap_{i\in I} \bigcap_{y_i \in Y_i} G_i(y_i)$. Hence we obtain

$$K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i\in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Theorem 3.1 ⇒ Lemma 2.2. For each $i \in I$, define a set-valued mapping $G_i\colon Y_i \to 2^X$ by

$$G_i(y_i) = X \setminus A_i^{-1}(y_i), \quad \forall \ y_i \in Y_i.$$

By the conditions (1)–(3) of Lemma 2.2 and the definition of G_i, it is easy to see that all conditions (1)–(3) of Theorem 3.1 are satisfied. By theorem 3.1, we have

$$K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i\in Y_i} G_i(y_i) \right) \neq \emptyset.$$

Taking any $\hat{x} \in K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i\in Y_i} G_i(y_i) \right)$, we have $\hat{x} \in K$ and

$$\hat{x} \in G_i(y_i) = X \setminus A_i^{-1}(y_i), \quad \forall \ y_i \in Y_i \text{ and } i \in I.$$

It follows that

$$y_i \notin A_i(\hat{x}), \quad \forall \ y_i \in Y_i \qquad \text{and } i \in I$$

and so $A_i(\hat{x}) = \emptyset$ for all $i \in I$. The conclusion of Lemma 2.2 holds. □

Remark 3.1 Theorem 3.1 generalizes Theorem 2.1 of Ding [21] from G-convex spaces to FC-spaces without any convexity structure. Theorem 3.1 also improves and generalizes Theorems 2 and 4 of Park and Kim [22] in several aspects and our argument method is completely different from that in [22].

Theorem 3.2 *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i\in I} X_i$ be the FC-space defined as in Lemma 2.1, and K be a nonempty compact subset of X. Let $G_i\colon X_i \to 2^X$ be such that for each $i \in I$,*

(1) *for each $N = \{x_0, \ldots, x_n\} \in < X >$ and each $M = \{x_{i_0}, \ldots, x_{i_k}\} \subseteq N$, $\varphi_N(\Delta_k) \subseteq \bigcup_{x\in M} G_i(\pi_i(x)))$,*
(2) *for each $x_i \in X_i$, $G_i(x_i)$ is closed in X,*
(3) *for each $N_i \in \langle X_i \rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap (X_i \setminus G_i^{-1}(x)) \neq \emptyset$.*

Then we have

$$K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i\in X_i} G_i(y_i) \right) \neq \emptyset.$$

Proof For each $i \in I$, let $Y_i = X_i$, $X = Y = \prod_{i\in I} X_i$, and F is the identity mapping on X. Then, it is easy to see that the the conclusion of Theorem 3.2 holds from Theorem 3.1. □

Corollary 3.1 *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and $X = \prod_{i\in I} X_i$ be the compact FC-space defined as in Lemma 2.1. Let $G_i\colon X_i \to 2^X$ be such that for each $i \in I$,*

(1) *for each $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and each $M = \{x_{i_0}, \ldots, x_{i_k}\} \subseteq N$, $\varphi_N(\Delta_k) \subseteq \bigcup_{x\in M} G_i(\pi_i(x))$,*
(2) *for each $x_i \in X_i$, $G_i(x_i)$ is closed in X.*

Then we have

$$\bigcap_{i\in I} \bigcap_{y_i\in X_i} G_i(y_i)) \neq \emptyset.$$

Proof For each $i \in I$ and $N_i \in \langle X_i\rangle$, let $L_{N_i} = X_i$ and let $K = \prod_{i\in I} X_i$, then $K = X$ is compact. Clearly the condition (3) of Theorem 3.2 is satisfied trivially. The conclusion of Corollary 3.1 holds from Theorem 3.2. □

Theorem 3.3 *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i\in I} X_i$ be the FC-space defined as in Lemma* 2.1, *and K be a nonempty compact subset of X. Let $G_i\colon X_i \to 2^X$ be such that for each $i \in I$,*

(1) *for each $x \in X$, $X_i \setminus G_i^{-1}(x)$ is a FC-subspace of X_i,*
(2) *for each $x \in X$, $\pi_i(x) \in G_i^{-1}(x)$,*
(3) *for each $x_i \in X_i$, $G_i(x_i)$ is closed in X,*
(4) *for each $N_i \in \langle X_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \bigcap (X_i \setminus G_i^{-1}(x)) \neq \emptyset$.*

Then we have

$$K \bigcap \left(\bigcap_{i\in I} \bigcap_{y_i\in X_i} G_i(y_i) \right) \neq \emptyset.$$

Proof We first show that the conditions (1) and (2) imply the condition (1) of Theorem 3.2 holds. If it is false, then there exist $i \in I$, $N = \{x_0, \ldots, x_n\} \in \langle X\rangle$ and $M = \{x_{i_0}, \ldots, x_{i_k}\} \subseteq N$ such that $\varphi_N(\Delta_k) \not\subseteq \bigcup_{x\in M} G_i(\pi_i(x))$. Hence there exists a $\hat{x} \in \varphi_N(\Delta_k) = \prod_{i\in I} \varphi_{N_i}(\Delta_k)$ where $N_i = \pi_i(N)$ such that $\hat{x} \notin G_i(\pi_i(x))$ for all $x \in M$. Hence $\pi_i(M) \subseteq X_i \setminus G_i^{-1}(\hat{x})$. Since $X_i \setminus G_i^{-1}(\hat{x})$ is a *FC*-subspace by (1), we have $\pi_i(\hat{x}) \in \varphi_{N_i}(\Delta_k) \subseteq X_i \setminus G_i^{-1}(\hat{x})$ and so $\pi_i(\hat{x}) \notin G_i^{-1}(\hat{x})$, which contradicts the condition (2). This show that the conditions (1) and (2) imply the condition (1) of Theorem 3.2 holds. The conclusion of Theorem 3.3 holds from Theorem 3.2. □

As a direct consequence of Theorem 3.3, we have

Corollary 3.2 *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and $X = \prod_{i\in I} X_i$ be the compact FC-space defined as in Lemma* 2.1. *Let $G_i\colon X_i \to 2^X$ be such that for each $i \in I$,*

(1) *for each $x \in X$, $X_i \setminus G_i^{-1}(x)$ is a FC-subspace of X_i,*
(2) *for each $x \in X$, $\pi_i(x) \in G_i^{-1}(x)$,*
(3) *for each $x_i \in X_i$, $G_i(x_i)$ is closed in X. Then we have.*

$$\bigcap_{i\in I}\bigcap_{y_i\in X_i} G_i(y_i)\neq\emptyset.$$

Remark 3.2 Noting that any convex subset of a topological vector space is a *FC*-space, Corollary 3.2 improves and generalizes Theorem 2.1 of Guillerme [12] from compact convex subsets of topological vector spaces to *FC*-spaces without any convexity structure under much weaker assumptions.

4 Existence of weak pareto equilibria

In this section, by using our nonempty intersection Theorems, we shall establish some existence theorems of weak Pareto equilibria for GMOG (1).

The following result is Theorem 14.18 of Aliprantis and Border [23].

Lemma 4.1 *Let X and Y be topological spaces and $G: X \to 2^Y$ be a set-valued mapping. Then the following statements are equivalent:*

(1) *G is lower semi-continuous at a point $x \in X$,*
(2) *if $x_\alpha \to x$, then for each $y \in G(x)$, there exists a subnet $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of the index set $\{\alpha\}$ and elements $y_\lambda \in G(x_{\alpha_\lambda})$ for each $\lambda \in \Lambda$ such that $y_\lambda \to y$.*

Theorem 4.1 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a FC-space, K be a nonempty compact subset of $X = \prod_{i\in I} X_i$, and Z_i be a topological space. Let $F_i\colon X^i \times X_i \to 2^{Z_i}$ and $D_i\colon X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *for each $N = \{y_0,\dots,y_n\} \in < X >$, $M = \{y_{i_0},\dots,y_{i_k}\} \subseteq N$, and each $x \in \varphi_N(\Delta_k)$, there exist $y \in M$ and $z_i \in F_i(x^i,x_i)$ such that $z_i - F_i(x^i,\pi_i(y)) \subseteq Z_i \setminus D_i(x)$,*
(2) *for each $y_i \in X_i$, the set $\{x \in X : \exists\, z_i \in F_i(x^i,x_i)$ such that $z_i - F_i(x^i,y_i) \subseteq Z_i \setminus D_i(x)\}$ is closed in X,*
(3) *for each $N_i \in <X_i>$, there exists a compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying*

$$(z_i - F_i(x^i,y_i))\bigcap D_i(x)\neq\emptyset,\quad \forall\, z_i \in F_i(x^i,x_i).$$

Then there exists $\hat{x} = (\hat{x}^i,\hat{x}_i) \in K$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i,\hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i,y_i))\bigcap D_i(\hat{x}) = \emptyset,\quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Proof For each $i \in I$, define a set-valued mapping $G_i\colon X_i \to 2^X$ by

$$G_i(y_i) = \{x \in X : \exists\, z_i \in F_i(x^i,x_i) \text{ such that} (z_i - F_i(x^i,y_i))\bigcap D_i(x) = \emptyset\},\quad \forall y_i \in X_i. \square$$

By the definition of G_i, it follows from the condition (1) that for each $N = \{y_0,\dots,y_n\}$ and $M = \{y_{i_0},\dots,y_{i_k}\}$, $\varphi_N(\Delta_k) \subseteq \bigcup_{y\in M} G_i(\pi_i(y))$. Hence the condition (1) of Theorem 3.2 holds. Clearly the condition (2) implies the condition (2) of Theorem 3.2 holds. It follows from the condition (3) that for each $N = \{y_0,\dots,y_n\}$ and $M = \{y_{i_0},\dots,y_{i_k}\} \subseteq N$, there exists compact *FC*-subspace L_{N_i} of X_i and for each

$x \in X \backslash K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying $(z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset$ for all $z_i \in F(x^i, x_i)$. By the definition of G_i, we have that for each $x \in X$,

$$G_i^{-1}(x) = \{y_i \in X_i : \exists z_i \in F_i(x^i, x_i) \text{ such that } (z_i - F_i(x^i, y_i)) \bigcap D_i(x) = \emptyset\}$$

and

$$X_i \backslash G_i^{-1}(x) = \{y_i \in X_i : (z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \ \forall\, z_i \in F(x^i, x_i)\}.$$

Hence we have

$$y_i \in L_{N_i} \bigcap (X_i \setminus G_i^{-1}(x))$$

and hence $L_{N_i} \bigcap (X_i \setminus G_i^{-1}(x)) \neq \emptyset$. The condition (3) of Theorem 3.2 is also satisfied. By Theorem 3.2, $K \bigcap \ (\bigcap_{i\in I} \ \bigcap_{x_i \in X_i} G_i(x_i)) \neq \emptyset$. Taking any $\hat{x} \in K \bigcap \ (\bigcap_{i\in I} \ \bigcap_{y_i \in X_i} G_i(y_i))$, then for each $i \in I$, there exists $z_i \in F_i(\hat{x}^i, \hat{x}_i)$ such that

$$(z_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \ \ \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Theorem 4.2 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a FC-space, K be a nonempty compact subset of $X = \prod_{i\in I} X_i$, and Z_i be a topological space. Let $F_i\colon X^i \times X_i \to 2^{Z_i}$ and $D_i : X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *for each $x \in X$, the set $\{y_i \in X_i : (z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \ \forall\, z_i \in F(x^i, x_i)\}$ is a FC-subspace of X_i,*
(2) *for each $x \in X$, there exists $z_i \in F_i(x^i, x_i)$ such that $(z_i - F_i(x^i, x_i)) \bigcap D_i(x) = \emptyset$,*
(3) *for each $y_i \in X_i$, the set $\{x \in X : \exists\, z_i \in F_i(x^i . x_i)$ such that $z_i - F_i(x^i, y_i) \subseteq Z_i \backslash D_i(x)\}$ is closed in X,*
(4) *for each $N_i \in <X_i>$, there exists a compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exist $i \in I$ and $y_i \in L_{N_i}$ satisfying*

$$(z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \quad \forall\, z_i \in F_i(x^i, x_i).$$

Then there exists $\hat{x} = (\hat{x}^i, \hat{x}_i) \in K$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \ \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Proof For each $i \in I$, define a set-valued mapping $G_i\colon X_i \to 2^X$ by

$$G_i(y_i) = \{x \in X : \ \exists\, z_i \in F_i(x^i, x_i) \text{ such that } (z_i - F_i(x^i, y_i)) \bigcap D_i(x) = \emptyset\}, \quad \forall y_i \in X_i.$$

By (1), for each $x \in X$, $X_i \backslash G_i^{-1}(x) = \{y_i \in X_i : (z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \ \forall\, z_i \in F(x^i, x_i)\}$ is a FC-subspace of X_i, the condition (1) of Theorem 3.3. By the definition of G_i and condition (2), we have that for each $x \in X$, $\pi_i(x) \in G_i^{-1}(x)$. The condition (2) of Theorem 3.3 holds. By the proof of Theorem 4.1, we know that the conditions (3) and (4) imply the conditions (3) and (4) of Theorem 3.3 hold. By Theorem 3.3,

$K \bigcap (\bigcap_{i \in I} \bigcap_{x_i \in X_i} G_i(x_i)) \neq \emptyset$. Taking any $\hat{x} \in K \bigcap (\bigcap_{i \in I} \bigcap_{y_i \in X_i} G_i(y_i))$, then for each $i \in I$, there exists $z_i \in F_i(\hat{x}^i, \hat{x}_i)$ such that

$$(z_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1). □

If each (X_i, φ_{N_i}) is a compact FC-space in Theorems 4.1 and 4.2, as direct consequences of Theorems 4.1 and 4.2, we easy obtain the following results.

Corollary 4.1 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and Z_i be a topological space. Let $F_i : X^i \times X_i \to 2^{Z_i}$ and $D_i : X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *for each $N = \{y_0, \ldots, y_n\} \in \langle X \rangle$, $M = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, and each $x \in \varphi_N(\Delta_k)$, there exist $y \in M$ and $z_i \in F_i(x^i, x_i)$ such that $z_i - F_i(x^i, \pi_i(y)) \subseteq Z_i \setminus D_i(x)$,*
(2) *for each $y_i \in X_i$, the set $\{x \in X : \exists\, z_i \in F_i(x^i.x_i)$ such that $z_i - F_i(x^i, y_i) \subseteq Z_i \setminus D_i(x)\}$ is closed in X.*

Then there exists $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Corollary 4.2 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and Z_i be a topological space. Let $F_i : X^i \times X_i \to 2^{Z_i}$ and $D_i : X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *for each $x \in X$, the set $\{y_i \in X_i : (z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \forall\, z_i \in F(x^i, x_i)\}$ is a FC-subspace of X_i,*
(2) *for each $x \in X$, there exists $z_i \in F_i(x^i, x_i)$ such that $(z_i - F_i(x^i, x_i)) \bigcap D_i(x) = \emptyset$,*
(3) *for each $y_i \in X_i$, the set $\{x \in X : \exists\, z_i \in F_i(x^i.x_i)$ such that $z_i - F_i(x^i, y_i) \subseteq Z_i \setminus D_i(x)\}$ is closed in X.*

Then there exists $\hat{x} = (\hat{x}^i, \hat{x}_i) \in K$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Corollary 4.3 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and Z_i be a topological space. Let $F_i : X^i \times X_i \to 2^{Z_i}$ and $D_i : X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *the mapping $W_i : X \to 2^{Z_i}$ defined by $W_i(x) = Z_i \setminus D_i(x)$ has closed graph on $X \times Z_i$.*
(2) *for each $N = \{y_0, \ldots, y_n\} \in \langle X \rangle$, $M = \{y_{i_0}, \ldots, y_{i_k}\} \subseteq N$, and each $x \in \varphi_N(\Delta_k)$, there exist $y \in M$ and $z_i \in F_i(x^i, x_i)$ such that $z_i - F_i(x^i, \pi_i(y)) \subseteq W_i(x)$,*
(3) *F_i is upper semicontinuous with nonempty closed values,*
(4) *for each $y_i \in X_i$, $x^i \mapsto F_i(x^i, y_i)$ is lower semicontinuous on X^i.*

Then there exists $\hat{x} = (\hat{x}^i, \hat{x}_i) \in X$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Proof Clearly, the condition (2) implies the condition (1) of Corollary 4.1 is satisfied. Now we show that the conditions (1), (3), and (4) imply the condition (2) of Corollary 4.1 holds. For each $i \in I$ and $y_i \in X_i$, let $G_i(y_i) = \{x \in X : \exists\, z_i \in F_i(x^i, x_i)$ such that $z_i - F_i(x^i, y_i) \subseteq W_i(x)\}$. Let $x_\alpha = ((x_\alpha)^i, (x_\alpha)_i) \in G_i(y_i)$ and $x_\alpha \to \hat{x} \in X$, then there exists $(z_\alpha)_i \in F_i(x_\alpha)$ such that

$$(z_\alpha)_i - F_i((x_\alpha)^i, y_i) \subseteq W_i(x_\alpha), \quad \forall\, \alpha.$$

Since X is compact, By (3) and the Proposition 3.1.11 of Aubin and Ekeland [24], $F_i(X)$ is also compact. Without loss of generality, we can assume $(z_\alpha)_i \to \hat{z}_i$. By (3), we have $\hat{z}_i \in F_i(\hat{x}) = F_i(\hat{x}^i, \hat{x}_i)$. Since for each $y_i \in X_i$, $x^i \mapsto F(x^i, y_i)$ is lower semi-continuous by (4), it follows from Lemma 4.1 that for each $u_i \in F_i(\hat{x}^i, y_i)$, there exists a subnet $\{\alpha_\lambda\}$ of the index net $\{\alpha\}$ and elements $(u_\lambda)_i \in F_i((x_{\alpha_\lambda})^i, y_i)$ such that $(u_\lambda)_i \to u_i$. But we have

$$(z_\lambda)_i - (u_\lambda)_i \in (z_\lambda)_i - F_i((x_{\alpha_\lambda})^i, y_i) \subseteq W_i(x_{\alpha_\lambda})$$

and $(z_\lambda)_i - (u_\lambda)_i \to \hat{z}_i - u_i$. By (1), we have $\hat{z}_i - u_i \in W_i(\hat{x})$. Hence have $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ and $\hat{z}_i - F_i(\hat{x}^i, y_i) \subseteq W_i(\hat{x})$. Therefore, $\hat{x} \in G_i(y_i)$ and for each $y_i \in X_i$, $G_i(y_i)$ is closed in X. The conclusion of Corollary 4.3 holds from Corollary 4.1. □

As a consequence of Corollary 4.3, we have the following result.

Corollary 4.4 *Let I be any (finite or infinite) set of players. For $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and Z_i be a topological space. Let $F_i : X^i \times X_i \to 2^{Z_i}$ and $D_i : X \to 2^{Z_i}$ be two set-valued mappings such that for each $i \in I$,*

(1) *the mapping $W_i : X \to 2^{Z_i}$ defined by $W_i(x) = Z_i \setminus D_i(x)$ has closed graph on $X \times Z_i$.*
(2) *for each $x \in X$, the set $\{y_i \in X_i : (z_i - F_i(x^i, y_i)) \bigcap D_i(x) \neq \emptyset, \forall\, z_i \in F(x^i, x_i)\}$ is a FC-subspace of X_i,*
(3) *for each $x \in X$, there exists $z_i \in F_i(x^i, x_i)$ such that $(z_i - F_i(x^i, x_i)) \bigcap D_i(x) = \emptyset$,*
(4) *F_i is upper semicontinuous with nonempty closed values,*
(5) *for each $y_i \in X_i$, $x^i \mapsto F_i(x^i, y_i)$ is lower semi-continuous on X^i.*

Then there exists $\hat{x} = (\hat{x}^i, \hat{x}_i) \in K$ such that for each $i \in I$, there is a $\hat{z}_i \in F_i(\hat{x}^i, \hat{x}_i)$ satisfying

$$(\hat{z}_i - F_i(\hat{x}^i, y_i)) \bigcap D_i(\hat{x}) = \emptyset, \quad \forall\, y_i \in X_i,$$

i.e., $\hat{x}$ is a weak Pareto equilibrium point of the GMOG (1).

Remark 4.1 Noting that each convex subset of a topological vector space is a *FC*-space, hence Corollary 4.4 improves and generalizes Theorem 2.1 of Luo [13] and Theorem 3.3 of Guillerme [12] in the following aspects: (1) from convex subsets of topological vector spaces to any *FC*-spaces without any convexity structure; (2) each Z_i may be any topological space; (3) each $D_i(x)$ may not have any cone-type structure. Corollary 4.4 also generalizes the corresponding results in [5–11] to generalized multiobjective games and to FC-spaces.

Acknowledgements This project was supported by the NSF of Sichuan Education Department of China (2003A081) and SZD0406.

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